Practical Noises and Pragmatism

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Practical Noises and Pragmatism

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Abstract

Investigating about any practical designation of elements specified by some property, one comes across the possibility of applying the philosophical notion of *pragmatism*. We here try to develop the pragmatist view on the set-theoretical analysis and treat practical remainders of conception along this line.

1. Pragmatist View

The term *pragmatism*[1)] has been introduced to denote the general tendency to subordinate logical thinking to the ends of practical life and to find the test of the truth of ideas in their practical consequences. In the study of mathematical foundations, if we take up the aggregate of elements which are found eligible through some practical procedures in comparison with the set which has beforehand been put forward by abstraction as the total universe containing the above aggregate, there may be expected some pragmatist view to give a valid line of discussion. Suppose that we have a total set of elements T provided with a certain testing device $\boldsymbol{P}$ such that if $a \in T$ and $\boldsymbol{P}(a)$ (that is, a is an element of T specifically qualified by $\boldsymbol{P}$) a is called a *practical element*, and T_P is the aggregate of all practical elements of T, then the part defined as $R(T_P) = T - T_P$, which we call the *practical remainder* of T (with respect to $\boldsymbol{P}$), will make a theoretical noise in the meaning that we should ask whether $R(T_P)$ is void or not. If we have no possitive way really to distinguish unpractical elements from practical ones in the construction of T, $R(T_P)$ is expressly called a *practical noise*.

In pragmatism, any conception, if apart from its practical consequences, should be regarded merely as an abstraction without meaning or significance and eventually without truth or falsehood[1)]. Under this view, if there is no way really to prove any element of T to be an unpractical element, $R(T_P)$ may be regarded as a void set. However, as a matter of fact, a pragmatist process of conclusion should not be a mere formal process of inference. So, it shall not be admissible if one, with no further discussions, asserts that any practical noise, in reality, gives no other object than a void set.

The term *pragmatization* is sometimes used to mean a conduct which represents what is imaginary or subjective as real or actual, or materializes it through some physical characterization. So it may also be said to be pragmatist if we assume a euclidian space (of finite dimension) to be filled up by some homogeneous

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medium (say, ether) everywhere equidensely. Then the a priori measure* of a set M in this space may be expected to coincide with the total weight of the medium which just fills the spatial occupation of M. The measure of a set so pragmatized is called the *pragmatist measure*. Consequently, we shall take the pragmatist measure to be construed as meaning the a priori mesure itself in the sense that the former is a pragmatization of the latter.

2. Practical Enumerability

If there is assumed a sequence of elements $X=(x_k)$ $(k=1,2,\cdots)$, X is regarded as an enumerable set. But, if we have no way, in advance, to distinguish x_k from other elements, the enumerability of X may only be an abstract conception, so that, if we apply pragmatism, X shall be thought meaningless. In this regard, we say 'X is *practically enumerable*' when x_k is distinguished from other elements and if

$$x_1, \cdots, x_k$$

are known the next element x_{k+1} is always exactly determined; and then X is called a *practical sequence*. The enumerability that Cantor assumed in his diagonal process was just the practical enumerability. However, when one simply assumes such a set M that

$$\overline{M}<\Omega \quad \text{(the initial ordinal number of the 3rd class)},$$

then M may not always be thought to be practically enumerable. If M is not given to be practically enumerable, M may be a mere abstract concept and so shall be thought to be meaningless.

If Λ is a simply ordered set and every element of a set X is denoted as $x_\lambda(\lambda\in\Lambda)$, and x is given as

$$x=\lim x_\lambda\,,$$

and if there is no sequence (λ_k) such that

$$x=\lim x_{\lambda_k}\,,$$

we may eventually think that there is no practical way to reach x by the elements of X, because no stepping of elevation can, by human approach, be realized beyond enumerability. Thus we find a pragmatist ground for andopting the following postulate in the course of our empiricist theory of analysis.

Postulate 1.** *If Λ is a simply ordered set and if*

$$x=\lim x_\lambda\,,$$

* The empiricist extension of the Lebesgue measure[2].

** This postulate has previously been used and played a key role in theories of a priori measure and of empiricist analysis.

then there is a practical sequence (λ_k) $(\subseteq \Lambda)$ *such that*

$$x = \lim x_{\lambda_k}.$$

3. Set of Real Numbers

We use the convention on the decimal expression that applies the infinite form

$$a = 0.a_1 \cdots a_{n-1}(a_n - 1)\,99\cdots$$

instead of the finite one

$$a = 0.a_1 \cdots a_n,$$

so that any real number which is not equal to zero may be promised a unique expression of the infinite decimal form. Thus, whenever we assume a positive real number $x \leqslant 1$, we may thake x to be uniquely expressed in the form

$$x = 0.x_1 x_2 \cdots.$$

However, if the sequence (x_k) cannot really be a practical one, x must be a mere abstract object and so, in our view, be meaningless. In this regard, we call x a *practical* (*real*) *number* if (x_k) is a practical sequence. If we denote as $I = (0, 1]\,(\equiv \{x :\ 0 < x \leqslant 1\})$ and by I_P the aggregate of all practical numbers in I, the remainder

$$R(I_P) = I - I_P$$

causes a practical noise.

We may as practical numbers show

$$\frac{3}{5},\ \pi = 3.14159\cdots,\ \tan^{-1}\frac{1}{3},\ \text{etc.},$$

but it is evident that practical numbers which are really to be shown by the human race can only make an enumerable set in all. However, it is also evident that we cannot wholly know what practical numbers will be found in the future. While, in our course of logic, the following proposition is proved:

Proposition 1 (*Theorem of Cantor*). I_P *cannot be practically enumerable.*

Proof. If I_P is practically enumerable, there must be a practical sequence $(\overset{(k)}{x})$ $(k = 1, 2, \cdots)$ such that

$$I_P = (\overset{(k)}{x}).$$

Then, as each $\overset{(k)}{x}$ are practical numbers, there are unique infinite decimal expressions

$$\overset{(k)}{x} = 0.\overset{(k)}{x_1}\,\overset{(k)}{x_2}\cdots.$$

Let the function $g(x)$ be defined as

$$g(x) = x+1 \text{ when } x<9\,,$$
$$\quad = x-1 \text{ when } x=9\,,$$

and let ξ_k be defined by $\xi_k = g(\overset{(k)}{x_k})$ $(k=1, 2, \cdots)$, then

$$\xi = 0.\xi_1\xi_2\cdots$$

gives a practical number which cannot be found in the sequence $(\overset{(k)}{x})$, whereas it is clear that

$$0<\xi\leqslant 1\,,$$

and hence

$$\xi \in I_P\,,$$

which gives a contradiction.

Now let us turn to the question of the practical noise $R(I_P)$. If we take x to belong to I, x is usually expected to be written in the form

$$x = 0.x_1x_2\cdots\,,$$

so that x may be expected as a pratical number. But, in our view, this cannot be assured when the sequence (x_k) is not found as a practical one. In any actual work, if $x \in I$ is assumed, x is either (i) really a practical number, or (ii) not yet practically shown. In case of (ii), if we need exactly to describe x, we may have the following two ways: (1) $x \fallingdotseq 0.x_1x_2\cdots x_n$, or (2) $0.x_1x_2\cdots x_n < x < 0.x_1x_2\cdots x_{n+1}$. In the case of (1) it is thought possible that $x = 0.x_1x_2\cdots x_n$, and in the case of (2), as a matter of fact, the interval $(0.x_1x_2\cdots x_n,\ 0.x_1x_2\cdots x_{n+1})$ is taken as the object of observation instead of the ghost point x. Such being the conditions, it may be said that our actual work is always tightly bound to practical numbers and hence $R(I_P)$ here causes no real obstruction. Consequently, we may say that $R(I_P)$ pragmatistly leaks no part of it really to effect, so that $R(I_P)$ may be regarded to be deletable from our course of analysis.

If a fact is, through our pragmatist view, demonstrated, we say that it is, *in empiricist pragmatism*, gained. Then, from the above, we have:

Proposition 2. *In empiricist pragmatism, it is admissible that* $R(I_P) = \varnothing$.

4. Quasi-Practical Space

Supposing that T is a metric space and $\tilde{T}_P$ the completion of T_P*, we, in this section, will examine what may happen when $R(\tilde{T}_P)$ $(= T - \tilde{T}_P) \neq \varnothing$.

We denote by $|x-z|$ the distance between x and z of T and call a set $S(z, r)$ defined as

$$S(z, r) = \{x:\ |x-z| = r\}$$

* The subspace of T which consists of all practical points (i.e., practical elements) of T.

a *sphere* in T. Then, if we define $B(z, r)$ by

$$B(z, r) = \bigcup_{\rho \leqq r} S(z, \rho),$$

it is evident that

$$(\forall r > 0)(B(z, r) \cap \tilde{T}_P \neq \varnothing). \Rightarrow . z \in \tilde{T}_P.$$

Therefore, if $z \in R(\tilde{T}_P)$, it follsws that

$$(\exists r_z > 0)(B(z, r_z) \cap \tilde{T}_P = \varnothing)$$

which implies that

$$B(z, r_z) \subseteq R(\tilde{T}_P).$$

Definition. If $R(\tilde{T}_P)$ does not contain any interior point, T is called a *quasi-practical space.*

Then, from the above discussion we conclude:

Proposition 3. *If T is a metric space, for T to be quasi-practical, it is necessary and sufficient that*

$$R(\tilde{T}_P) = \varnothing,$$

that is,

$$T \subseteq \tilde{T}_P.$$

If it is unprovable that $R(T_P) \neq \varnothing$, then T is called a *densely computable set* (with respect to $\boldsymbol{P}$). The case $R(T_P) = \varnothing$ gives an instance of such a T.

Proposition 4. *If T is a metric space and is densely computable, then T is necessarily quasi-practical.*

Demonstration. If there exist a point z and a positive real number r_z such that $B(z, r_z) \subseteq R(\tilde{T}_P)$, then it follows that $R(T_P) \neq \varnothing$. Then, the apodosis part is readily obtained.

5. Family of Borel sets

Let the total aggregate of ordinal numbers of the 2nd class be denoted by S. Then, in regard to Postulate 1, S cannot be treated as a determinate set, because S cannot be finished by any enumerable stepping in terms of its segments, whereas the construction of S may not be obtained without the stepping increase of orders of its elements. Thus we have the following conviction to be valid.

Proposition 5 (*Fundamental Claim in the Empiricist Analysis*). *If a set is given its ordinal to be of the 3rd class with no other specification to look into it, it is of an unfinished collection and cannot be regarded as a determinate set.*

A family $\boldsymbol{B}$ is the family of Borel sets if making use of ordinal numbers $\boldsymbol{B}$ can be classified in classes $\boldsymbol{B}_\alpha$, where $\alpha < \Omega$ (the initial ordinal of the 3rd class), in the following manner:

(i) the class $\boldsymbol{B}_0$ is the family of all closed sets;

(ii) if $\alpha=\lambda+n$, λ being a limit ordinal, and if n is an odd integer, B_α is the family of all sets of the form

$$\bigcup_{n=1}^{\infty} X_n ;$$

(iii) if n is an even integer, B_α is the family of all sets of the form

$$\bigcap_{n=1}^{\infty} X_n ;$$

$X_1, X_2, \cdots$ in (ii) or (iii) are sets which belong to classes of indices smaller than α.

According to Proposition 5, the family $\boldsymbol{B}$ must be of an unfinished collection. However, any set X of $\boldsymbol{B}$ may be regarded as a practical object if an index α is distinctly given such as $X \in \boldsymbol{B}_\alpha$, because then X may be reconstructed by enumerable compositions of union-makings and product-makings from certain sets of the class $\boldsymbol{B}_0$. Conversely, any practical set of $\boldsymbol{B}$ may reasonably defined as a set which is produced by enumerble compositions of union-makings and product-makings from sets of $\boldsymbol{B}_0$ and so belongs to a class $\boldsymbol{B}_\alpha$ for which α is distinctly determined. If X is simply assumed to be a set of $\boldsymbol{B}$ and the index of the class containing X is not settled, then X is only an abstract object and, in our pragmatist view, is to be cosidered meaningless. Thus we conclude:

Proposition 6. *The family of Borel sets is of an unfinished collection. However, it can be regarded densely computable and so, as a matter of fact, can be regarded to have no practical noise.*

In the above the initial class $\boldsymbol{B}_0$ is not discussed in detail, but it is pragmatistly possible to show that $\boldsymbol{B}_0$ may also be considered to have no practical noise.

6. On Probabilism

In this section, sets of points are uniformly restricted within a euclidian space of finite dimension $\boldsymbol{E}$ and every point of $\boldsymbol{E}$ is assumed to be equi-probable, that is, occurrences of any two points of $\boldsymbol{E}$ are always reckoned to be of equal probability. The probability that an aleatory variable point x, which is restricted within a given set K, occurs in a subset M of K, has usually been defined by

$$\Pr(x \in M \subseteq K) = \widetilde{m}M / \widetilde{m}K , \tag{6. 1}$$

where $\widetilde{m}M$ means the a priori measure of M and K is rather assumed to be a simple-figured set of which the measure value $\widetilde{m}K$ is evidently known (e. g., interiors of a sphere or a rectangle). In this case the practicality of Pr will naturally hinge upon the practicality of $\widetilde{m}M$, and so, practical sets in respect to Pr shall coincide with practical sets in respect to $\widetilde{m}$.

Stochastic books define Pr by another approach. They first take a so-called random sample

$$x_1, x_2, \cdots, x_N \tag{6. 2}$$

from within the set K, and then, if

$$x_{k_1}, x_{k_2}, \cdots, x_{k_{J(N)}}$$

are all of its points which belong to M, they consider the ratio

$$J(N)/N$$

to be an approximation of Pr by means of the sample (6.2). On this line Pr is evidently defined by

$$\Pr(x \in M \subseteq K) = \lim_{N \to \infty} J(N)/N.$$

This way of definition may literally be very practical, because its process of approaching is made up through an accumulation of actual practices of examinations. However, if we thereof make a theoretical inspection, we forthwith find an important omission, that is, the omission of assurance for the unique convergence of the sequence $(J(N)/N)$ $(N=1, 2, \cdots)$. In this context, there are undeniably found various sequences $(J(N)/N)$ which give several conditions of convergence, and the possibility of such severalty may reasonably be explained by the relative formula

$$\Pr(x \in M \subseteq K, \omega) = \int_{x \in M} \omega(x)\, dx \Big/ \int_{x \in K} \omega(x)\, dx \tag{6.3}$$

the left-hand of which means the relative probability of $M \subseteq K$ in respect to the weight function ω. The random-sampling approach is considered to be defined as an approach which corresponds to the case $(\forall x)(\omega(x)=1)$ in (6.3). Hence the practical noise of this case will hinge upon the noise about the practical possibility of the uniform event

$$(\forall x \in K)(\omega(x) = 1). \tag{6.4}$$

But, in effect, it is found very complicate to enforce (6.4).

Finally, there may be pointed out a physicalistic line to translate Pr as the mean density of the medium occupying M amomg K. In this connection, discussions inevitably regress toward the constructive formula (6.1) on the interpretation that $\tilde{m}$ means the pragmatist measure. Our investigation thus shall wholly be focussed on the subject of $\tilde{m}$-measurability.

7. Pragmatist $\tilde{m}$-Measurability

A determinate set M in a space $\boldsymbol{E}$ may generally be characterized by the property

$$(\forall x \in \boldsymbol{E})(x \in M . \vee . x \notin M). \tag{7.1}$$

However, in pragmatism, some notion so to say, of a *practical* (or *practically determinate*) *set* will be needed in addition. A partial characterization of such a set may be given by the following.

Criterion P. *If M is a practical set, M has no property which is essentially considered meaningless.*

In Section 1 we introduced the pragmatist $\tilde{m}$-measure to be put forward as a pragmatist interpretation of the $\tilde{m}$-measure. If we simply stand on this viewpoint, any set non-$\tilde{m}$-measurable may be thought to be a meaningless object, because any set having a determinate occupation in (a euclidian space) $\boldsymbol{E}$ must have a unique $\tilde{m}$-value (or $\tilde{m}$-measure value) to be determined as the total weight of the medium just filling its sptaial occupation. However, such a view seems as too much trusting to the physicalistic conviction on the phenomenal construction of a (euclidian) space. So, in this section, we try mathematically to develop a rather detailed course of discussions.

If a proposition $\sim p$ cannot be considered to be true, we say p is destined. For instance, if the relation

$$\tilde{m}M=0$$

is not considered to hold, then it shall be destined that

$$\tilde{m}M>0\,. \tag{7.2}$$

In this case, M is not necessarily promised to be $\tilde{m}$-measurable. (7.2) only means that the total weight of the medium filling the spatial occupation of M must be estimated to be positive.

However, in case of (7.2), if M is a practical set, there must be at least one $\tilde{m}$-measurable subset N in M such that

$$\tilde{m}N>0\,,$$

because, if there is no such N, the destination (7.2) can be no other than a mere abstract assumption and so only meaningless. Thus we may, by Criteion P, conclude :

Proposition 7 (*Principle of Null-Measure Assertion*). *If M is a practical set and*

$$(\forall X\subseteq M)\ (X \textit{ is measurable.}\Rightarrow .\tilde{m}X=0)\,,$$

then it must be that M is $\tilde{m}$-measurable and

$$\tilde{m}M=0\,.$$

We may really decide $\tilde{m}$-measures for many sets of points in $\boldsymbol{E}$ and such sets may directly be thought as practical ones. But whether there is a practical non-$\tilde{m}$-measurable set or not is not yet evidently settled. We will here, on setting the following axiom, look into this question.

Axiom P. *If M_k $(k=1, 2, \cdots)$ are all practical sets, then sets*

$$M_k-M_j,\ \ \cup M_k,\ \textit{and}\ \cap M_k$$

are practical.

Proposition 8. *If sets*

$$M_1 \subseteq M_2 \subseteq \cdots$$

all are $\widetilde{m}$-measurable and

$$M = \cup M_k ,$$

then M is $\widetilde{m}$-measurable and

$$\widetilde{m}M = \lim \widetilde{m}M_k . \tag{7.3}$$

Demonstration. If (7.3) is not assured, apparently it must be destined that

$$\widetilde{m}M > \lim \widetilde{m}M_k , \tag{7.4}$$

so that there exists a positive real number δ such that

$$\widetilde{m}(M - M_k) > \delta$$

is destined for all $k = 1, 2, \cdots$. Then, since

$$M - M_1 \supseteq M - M_2 \supseteq \cdots ,$$

for the set $N = \cap (M - M_k)$ it must be destined that

$$\widetilde{m}N \geqslant \delta . \tag{7.5}$$

Since N is a void set, (7.5) cannot be destined, so that the assumption (7.4) should eventually be a mere abstract one (and so meaningless). Then, by Axiom P and Criterion P, we have (7.3) to be left as the only case.

When M is a practical set, it may readily be seen that there exists a sequence of $\widetilde{m}$-measurable (and hence practical) sets (M_k) such that

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M$$

and any practical subset of $N = M - \cup M_k$ can have no other $\widetilde{m}$-value than zero. Then, by Proposition 7, N is $\widetilde{m}$-measurable and $\widetilde{m}N = 0$, and, by Proposition 8, we have $\widetilde{m}(\cup M_k) = \lim \widetilde{m}M_k$. Hence $\widetilde{m}M$ consequently is decided. Thus we have:

Proposition 9. *Any practical set is $\widetilde{m}$-measurable.*

Now we may say that we here are in the place to take up the problem of the noise between the family of determinate sets and that of practical sets. It seems apparently possible to delete the noise away. But we will here simply note that the formula (7.1) itself may not be considered as a complete expression, in some more stringent sense of philosophy.

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References

1) E.g., *Encyclopedia Americana.*
2) Kinokuniya, Y.: Mem. Muroran Inst. Tech. **6** (2), 492–494.